Impedance.

An electric circuit may consist of other network elements than resistors and sources, that we have considered until now. Such other elements does not in general have a simple proportional, time-independent relationship between current and voltage like in Ohm's law, $U(t) = RI(t)$. One basic element we are going to consider is the capacitor. It is characterised by its capacitance C and fulfil

$$I = C\frac{dU}{dt}. \tag{1}$$

Another basic element is the coil of self inductance L, which fulfil

$$U = L\frac{dI}{dt}. \tag{2}$$

However in the case of a harmonic varying voltage (or current) signal the corresponding current (or voltage) signal must be harmonic too. This opens up a generalisation of the concept of resistance in the case of alternating currents if phase shifts are taken into due account. For the capacitor we have that

$$U(t) = U_0 \cos(\omega t + \phi_U) \tag{3}$$

implies

$$I(t) = -\omega C U_0 \sin(\omega t + \phi_U) = \omega C U_0 \cos(\omega t + \phi_U + \frac{\pi}{2}). \tag{4}$$

This shows - as anticipated - that the current is harmonic varying like the voltage. However current is heading by 1/4 of a period. This is reasonable from a physical point of view since current has to flow for some time before building up an amount of charge in the capacitor creating a voltage difference between its conducting plates. If we want to write the current in the form of

$$I(t) = I_0 \cos(\omega t + \phi_I) \tag{5}$$

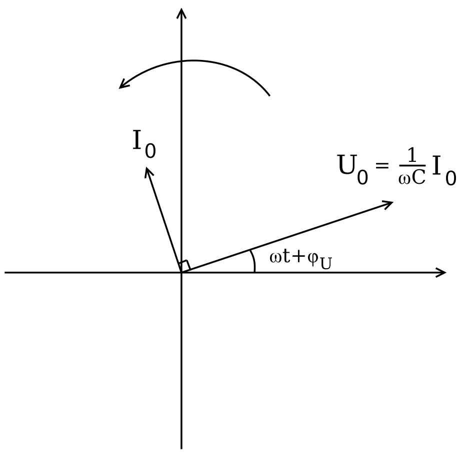
we should put

$$I_0 = \omega C U_0 \tag{6}$$

and

$$\phi_I = \phi_U + \frac{\pi}{2} \tag{7}$$

Although Ohm's law is not valid for the instantaneous values of U and I, we recognise from (6) that it holds for the amplitudes with a "resistance" being $1/\omega C$.



Figur 1 The phasor diagram of a capacitor.

Using g complex numbers we may include the phase shift $\frac{\pi}{2}$ in a simple way. We know that cosine can be expressed by the real part of the complex exponential function. Thus

$$U(t) = U_0 \cos(\omega t + \phi_U) = \mathrm{Re}\{U_0 \exp(i(\omega t + \phi_U))\} = \mathrm{Re}\{\hat{U} \exp(i\omega t)\} \tag{8}$$

where *the complex amplitude* $\hat{U}$ is defined in terms of the real amplitude U_0 and the phase shift ϕ_U as

$$\hat{U} = U_0 \exp(i\phi_U) \tag{9}$$

In the same manner we have

$$I(t) = I_0 \cos(\omega t + \phi_I) = \mathrm{Re}\{\hat{I} \exp(i\omega t)\} \tag{10}$$

with

$$\hat{I} = I_0 \exp(i\phi_I) \tag{11}$$

Furthermore if we define the complex *impedance* as

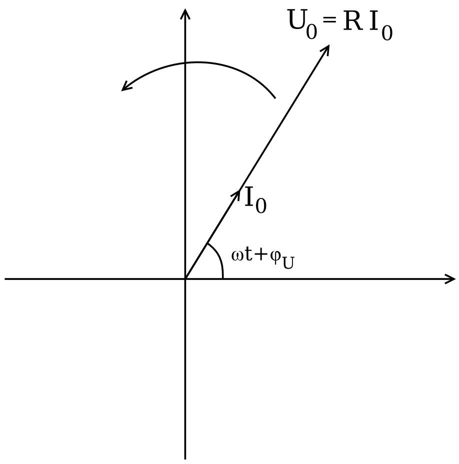
$$\hat{Z} = \frac{1}{i\omega C} = \frac{1}{\omega C} \exp(-i\frac{\pi}{2}) \tag{12}$$

we can put together the equations (6) and (7) into the *generalised Ohm's law*

$$\hat{U} = \hat{Z}\hat{I}. \tag{13}$$

Complex multiplication is exactly what we need since it consists in multiplication of moduli (amplitudes) and addition of arguments. Impedance has the unit (Ω) like ordinary resistance. The phasor diagram of figure 1 illustrates the amplitude- and phase relations between current and voltage of a capacitor.

For an ordinary Ohmic resistor,R is $\hat{Z} = R$, there is no phase shift between current and voltage. This is illustrated in figure 2.



Figur 2 The phasor diagram of a resistor.

Now - with the complex tool in hand - we are able to deduce the impedance of a self inductance along another line. We assume that currents and voltages are complex knowing that in the physical world they are only given bye the real part of these quantities. This works because we only deal with linear equations containing real coefficients. Thus the current is $I(t) = \hat{I}\exp(i\omega t)$. Inserting this into (2) we get

$$U(t) = i\omega L\hat{I}\exp(i\omega t) \tag{14}$$

If this is supposed to have the form $U(t) = \hat{U} \exp(i\omega t)$, we must have $\hat{U} = i\omega L\hat{I}$ and consequently

$$\hat{Z} = i\omega L = \omega L \exp(i\frac{\pi}{2}) \tag{15}$$

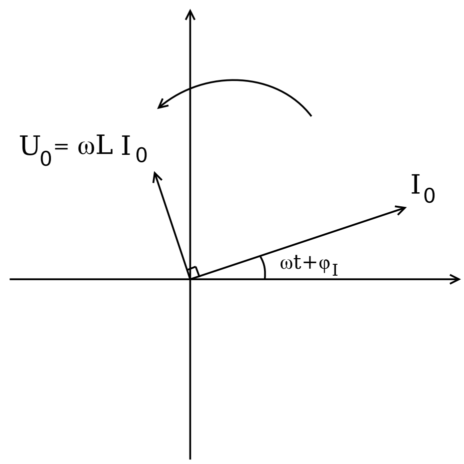
Since $|\exp(i\phi_U)| = 1$ we have $|\hat{U}| = |U_0 \exp(i\phi_U)| = U_0|\exp(i\phi_U)| = U_0$. Likewise $|\hat{I}| = I_0$ and $|\hat{Z}| = |i\omega L| = |i|\omega L = \omega L$. The modulus of equation (13) gives

$$U_0 = |\hat{U}| = |\hat{Z}\hat{I}| = |\hat{Z}||\hat{I}| = \omega L I_0 \tag{16}$$

Furthermore we have $\arg(\hat{U}) = \arg(U_0 \exp(i\phi_U)) = \arg(U_0) + \arg(\exp(i\phi_U)) = 0 + \phi_U = \phi_U$. The argument of (13) thus gives

$$\phi_U = \phi_I + \frac{\pi}{2} \tag{17}$$

We recognise that for the self inductance the voltage is 1/4 of a period ahead of the current. This is probably most easy to remember if we think in the mechanical analogy where self inductance corresponds with mass, voltage with force and current with velocity. A force has to be applied for some time before momentum and thereby velocity has been builded up. The phasor diagram of a self inductance is found in figure 3.



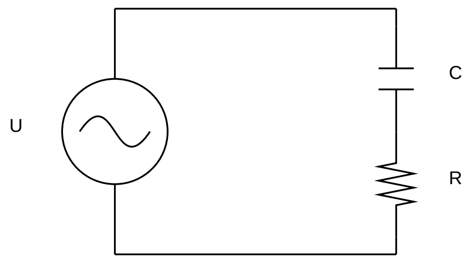
Figur 3 The phasor diagram of a self inductance.

The concept of impedance can also be applied to circuits composed of simple elements. Kirchhoff's laws are linear and the artifice of using complex currents and voltages during calculations taking the real part at the end is thus feasible. Since Kirchhoff's laws still applies we conclude that *impedances in series are additive.* However for impedances in parallel its the reciprocal impedances that should be added. It is thus convenient to introduce the concept of *admittance* by

$$\hat{Y} = \frac{1}{\hat{Z}} \tag{18}$$

The we may say that *admittances in parallel are additive.*

There is a certain mental threshold to be overcome in order to be familiar with this formalism; but the advantage of it can't be overrated. The equations describing a linear electric (or physical) network are in general a coupled set of integro-differential equations (in the time domain). If one only consider harmonic functions the equations becomes algebraic instead (in the frequency domain). This is no restriction in the study of the general dynamic behaviour of a physical system since any input-signal can be expressed as a weighed sum (or integral) of harmonic inputs of different frequencies. The linearity ensures that the total output is the sum harmonic outputs with the same weights.



Figur 4 Sine wave generator in series with a resistor and a capacitor.

Example 1: A resistor R and a capacitor C is connected in series as shown in figure (4) with a voltage source $U = U_0 \cos(\omega t)$. The starting point of time is chosen such that $\phi_U = 0$.
1) Determine the current amplitude I_0 and the phase shift ϕ_I of the current $I = I_0 \cos(\omega t + \phi_I)$.
The current may also be decomposed into a component I_A *in phase* and a component I_B *out of phase* with the the voltage in this way,

$$I(t) = I_A \cos(\omega t) + I_B \sin(\omega t) \tag{19}$$

2) Determine I_A and I_B.

3) Draw the phasor diagram.
Re 1) The impedance of the series connection becomes

$$\hat{Z} = \frac{1}{i\omega C} + R$$

resulting in the admittance

$$\hat{Y} = \frac{1}{\hat{Z}} = \frac{1}{\frac{1}{i\omega C} + R} = \frac{i\omega C}{1 + i\omega RC}$$

In general we have $I_0 = |\hat{Y}|U_0$ and $\phi_I = \phi_Y + \phi_U$. Proof:

$$\begin{aligned}
I(t) &= I_0 \cos(\omega t + \phi_I) = \mathrm{Re}\{I_0 \exp(i(\omega t + \phi_I))\} \\
&= \mathrm{Re}\{I_0 \exp(i\phi_I)) \exp(i\omega t)\} = \mathrm{Re}\{\hat{I} \exp(i\omega t)\} \\
&= \mathrm{Re}\{\hat{Y}\hat{U} \exp(i\omega t)\} = \mathrm{Re}\{|\hat{Y}| \exp(i\phi_Y) U_0 \exp(i\phi_U) \exp(i\omega t)\} \\
&= \mathrm{Re}\{|\hat{Y}|U_0 \exp(i(\omega t + \phi_Y + \phi_U))\} = |\hat{Y}|U_0 \,\mathrm{Re}\{\exp(i(\omega t + \phi_Y + \phi_U))\} \\
&= |\hat{Y}|U_0 \cos(\omega t + \phi_Y + \phi_U)).
\end{aligned}$$

Now

$$|\hat{Y}| = |\frac{i\omega C}{1 + i\omega RC}| = \frac{|i\omega C|}{|1 + i\omega RC|} = \frac{\omega C}{\sqrt{1 + (\omega RC)^2}}$$

and thus

$$I_0 = \frac{\omega C}{\sqrt{1 + (\omega RC)^2}} U_0$$

Furthermore

$$\phi_I = \phi_Y = \frac{\pi}{2} - \arg(1 + i\omega RC) = \frac{\pi}{2} - \arctan(\omega RC)$$

One deduces that the resistance determines the current at high frequencies:

$$|\hat{Y}| \to \frac{1}{R} \quad \text{og} \quad \phi_I \to 0 \quad \text{for} \quad \omega \to \infty$$

and that the capacitor determines the current at low frequencies:

$$|\hat{Y}| \to \omega C \quad \text{og} \quad \phi_I \to \frac{\pi}{2} \quad \text{for} \quad \omega \to 0$$

Re 2) From the expression $I(t) = \mathrm{Re}\{\hat{Y} U_0 \exp(i\omega t)\}$ another approach is to split $\hat{Y}$ into real and imaginary parts instead of using modulus and argument - the polar form - we did above,

$$\begin{aligned}
I(t) &= U_0 \,\mathrm{Re}\{(Y' + iY'')(\cos(\omega t) + i\sin(\omega t))\} \\
&= U_0 \,\mathrm{Re}\{(Y' \cos(\omega t) - Y'' \sin(\omega t)) + i(Y' \sin(\omega t) + Y'' \cos(\omega t))\} \\
&= U_0(Y' \cos(\omega t) - Y'' \sin(\omega t))
\end{aligned}$$

Hereby

$$I_A = U_0 Y' \quad \text{og} \quad I_B = -U_0 Y''$$

Y' and Y'' are found in this way:

$$\hat{Y} = \frac{i\omega C}{1 + i\omega RC} = \frac{i\omega C(1 - i\omega RC)}{1 + (\omega RC)^2} = \frac{\omega^2 RC^2}{1 + (\omega RC)^2} + i\frac{\omega C}{1 + (\omega RC)^2}$$

That is

$$Y' = \frac{\omega^2 RC^2}{1 + (\omega RC)^2} \quad \text{og} \quad Y'' = \frac{\omega C}{1 + (\omega RC)^2}$$