

# Ohms Law and its Analogues.

We are going to collect a number of building blocks that can be used in the modeling of physical systems. The first were a source that came in two forms, the effort (or voltage) source and the flow (or current) source. These elements are active and may deliver energy into a system. The third element consumes always energy and is called a *sink*.

*Electric currents.* Sinks are recognized in an electrical circuit as resistors. A resistor,  $R$  has according to Ohms law a time-independent proportionality between current,  $I$  and voltage drop,  $U$ .

$$U = RI$$

We will dwell a bit on the quantities entering Ohms law. Electric current is motion of charged particles. This was already speculated upon before one had a more detailed insight to the nature of these particles. Thus one imagined that the current passed from plus to minus in a wire connected to a galvanic element, believing that the moving particles were positive. However in a solid it is usually a current of negative electrons, that moves in the opposite direction. This holds for metals. Nevertheless one sticks to the convention that the current flows from plus to minus since in most cases one can't tell the difference of having positive particles moving in one direction or negative particles moving in the opposite direction. Electric charge has the unit Coulomb. This is a huge amount of electrons since the electron charge is  $e = 1.602 \cdot 10^{-19}C$ . Electric current has the unit Ampere, which is one Coulomb per second.

The electric potential is associated with the concept of potential energy. A (test) charge,  $q$  placed at  $\mathbf{r}$  in a space where other charges reside feels the forces of these other charges. It is thus able to conduct work. The work it is able to do is called the potential energy  $E_{pot}(\mathbf{r})$ . Since the magnitude of this potential energy is proportional to the magnitude of the (test) charge it is meaningful to introduce the potential,  $\phi(\mathbf{r})$ , which is the potential energy pr.

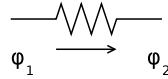
charge. Thus one has

$$E_{pot}(\mathbf{r}) = q\phi(\mathbf{r})$$

The unit of the potential is thus Joule per Coulomb,  $J/C$ . This unit is better known as Volt,  $V$ .

In order to get the right sign in Ohm's law one has to introduce an orientation rule (see Figure 1). Draw an arrow besides the resistor. This arrow indicates the direction of a positive current. If the actual current flows in the opposite direction this is just reflected in a negative  $I$ . The potential drop  $U = \phi_1 - \phi_2$  is the difference between the potential  $\phi_1$  at the tail of the arrow and the potential  $\phi_2$  at the head of the arrow. Notice that the potential drop is minus the potential increment,  $\Delta\phi = \phi_2 - \phi_1$ ,

$$U = -\Delta\phi$$



**Figure 1** The relation between potential,  $\phi$  og voltage drop,  $U$ :  $U = -\Delta\phi = \phi_1 - \phi_2$ .

The resistance  $R$  is specified in Ohm,  $\Omega$ . The resistance is both dependent on the material of the conductor but also on geometrical factors. For a wire of length  $l$  and a constant cross section  $A$  the resistance becomes,

$$R = \rho \frac{l}{A}$$

where  $\rho$ , the resistivity is the material parameter to be found in a data sheet. Often its reciprocal, the specific conductivity,  $\sigma = 1/\rho$  is found.

Exercise: A material of specific resistance  $\rho$  lies between two cylinders of radii  $r_1$  og  $r_2$  and length  $l$ . The cylindrical surfaces are highly conducting electrodes. What is the resistance of the material between the two electrodes?

Exercise: As above; but now between two spherical surfaces?

Introducing the specific resistance Ohm's law can be written,

$$U = \rho \frac{l}{A} I$$

or,

$$-\frac{\Delta\phi}{l} = \rho \frac{I}{A}$$

The potential drop per length,  $-\frac{\Delta\phi}{l}$  is called the electric field, denoted  $E$ . It has the unit  $V/m$ . The current per area,  $\frac{I}{A}$  is called the current density, denoted  $j$ . It has the unit  $A/m^2$ . Both quantities have a direction in space; they are vectors. Thus we can reformulate Ohm's law,

$$\mathbf{E} = \rho \mathbf{j} \text{ eller } \mathbf{j} = \sigma \mathbf{E}$$

The advantage of this formulation is that none of the quantities entering the equation refers to the geometry of the conductor.

Exercise: Explain that the force  $\mathbf{F}$  on a charge  $q$  is  $q\mathbf{E}$ . (Hint: What is the relationship between force and potential energy?)

*Thermal currents.* Conduction of heat can be described by the heat current density  $\mathbf{j}_h$  expressed in  $W/m^2$ . This current is proportional to the temperature gradient  $\nabla T$  by *Fourier's law*

$$\mathbf{j}_h = -\lambda \nabla T$$

$\lambda$  is called the specific heat conductivity. One sees that Fourier's law is completely analogous with Ohm's law, since  $-\nabla T$  tells how much temperature drops per length in the direction of the heat current, whereas the electric field denoted the potential drop per length.

Exercise: An outer wall is covered by insulating material in order to reduce heat losses. Across which layer the temperature drop is the largest?

Exercise: Find the heat current density as a function of radius through the insulation around a central heating tube. Find also the temperature profile.

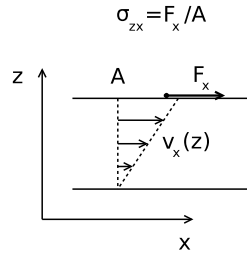
Exercise: A spherical heat source of radius  $r$  is placed in an infinite medium of specific heat conductivity  $\lambda$ . The source produces a constant power  $P$ . How large is the temperature at the source compared to the temperature in the far distance from the source?

*Fluid flow.* A flowing liquid loses energy by friction between parts of the liquid having different velocities. Kinetic energy is by this process converted into heat; but pressure gradients supply new energy allowing a stationary flow.

The friction is quantified by a liquid property called the viscosity  $\eta$ . Imagine a stationary flow between two large parallel plates a little distance apart. The plates are normal to the  $z$ -axis, the lower plate at rest while the upper plate is moved at a constant velocity in the  $x$ -direction. This creates a linear velocity-profile  $v_x(z)$ . It turns out that for many liquids (so-called) Newtonian) it takes a constant force  $F_x$  to maintain the flow. Furthermore the force is proportional to the area  $A$  and inversely proportional to the plate distance. The force per area is called the tangential stress  $\sigma_{zx}$ . (The index  $z$  indicates the surface being perpendicular to the  $z$ -axis, and the  $x$  indicates the force along the  $x$ -axis.) One has thus,

$$\sigma_{zx} = \eta \frac{\partial v_x}{\partial z}$$

This is once more an analogue with Ohm's law.



**Figure 2** Laminar flow of a Newtonian liquid. The tangential stress,  $\sigma_{zx}$  is proportional to the velocity gradient  $\frac{\partial v_x}{\partial z}$ .

The velocity-profile becomes different of the liquid flows through a cylinder tube. We expect the volume current  $\dot{V}$  to be proportional to the driving pressure drop  $-\Delta p$ , and inversely proportional to the length  $l$  of the tube and the viscosity,  $\eta$ . Dimensional reasoning hence give that  $\dot{V}$  is proportional to the 4th power of the radius  $r$ .

Exercise: Carry out the dimensional argument.

A detail calculation shows that the velocity profile becomes parabolic, the speed being at maximum in the middle of the tube. The mean velocity thus grows proportional to the radius  $r$ , and since  $\dot{V}$  is the product of the mean velocity and the cross section area the physics of the 4th power can be understood. The detailed calculation determines the proportionality factor and the result is called the *Poiseuille's law*,

$$\dot{V} = -\Delta p \frac{\pi r^4}{8\eta l}$$

If we define the fluid resistance,  $R_f = -\Delta p/\dot{V}$  we have,

$$R_f = \frac{8\eta l}{\pi r^4}$$

Exercise: Let the velocity-profile be  $v_z(r)$ . Find the forces acting on a thin cylindrical shell between radii  $r$  and  $r + dr$ . Use this in formulating a differential equation for  $v_z(r)$ . Solve this equation. What are the boundary conditions? Find the volume flow and thus Poiseuille's law.

The viscosity we also meet in the formular for the frictional force,  $F$  on a sphere with radius  $r$  that is driven through a liquid at velocity  $v$ . That is the so-called *Stoke's formular*

$$F = 6\pi\eta r v$$

Thus the mechanical resistance  $R_m = F/v$  is  $6\pi\eta r$ .

Exercise: A sphere immersed in a liquid is dropped and it begins to fall. Explain why the velocity grows to a maximum velocity  $v_\infty$ , and determine this. Find a characteristic time scale for the growth of the velocity from zero to  $v_\infty$ . Find the velocity as a function of time,  $v(t)$ .