Linear Response Theory.

The response theory deals with the general relationship between input $\phi(t)$ and output $\gamma(t)$ to a physical system.

Selection of variables.

The energetic interactions of a physical system with its surroundings are called *energy bonds*. Every energy bond is associated with two so-called conjugated variables, the generalised voltage, *effort*, e and the generalised current, *flow*, f. The product $e \cdot f$ should be some kind of transferred energy per time. The flow-variable is recognised by the feature that it changes sign under time reversal whereas the effort variable is invariant. In addition one defines two supplementary variables: the *displacement* q, which is the flow integrated over time, and the *momentum* p, which is the effort integrated over time.

Describing systems in this framework makes it easy to see analogies between different physical systems. We mention here a number of examples. In particle mechanics force, F and velocity, v are conjugated variables. In continuum physics (fluids or solids) they may be taken as the normal stress, σ and the volume flow, J_V. In electric energy bonds e corresponds to voltage, U, f to current, I, q to charge, Q and p to magnetic flux, Φ_B. The conjugated variables in a thermal energybond can be chosen as the Carnotfactor, $1-T_0/T$ [1] and the heat current, J_Q. In this case the product is the free energy. One may alternatively chose e as temperature, T and f as entropy current, J_S.

Linear systems with memory.

In linear response theory it is assumed that for sufficiently small actions there will be a linear relationship between input, ϕ and output, γ; but there may be an "after effect". The change $d\phi(t')$ in ϕ at time t' gives thus a contribution to the change $d\gamma(t)$ in γ at time t:

$$d\gamma(t) = R(t,t')d\phi(t') = R(t-t')d\phi(t'). \tag{1}$$

Here it has been assumed that this change is only dependent on the time difference (homogeneity of time). Since response follows stimuli we have that

1 Which amounts to the relative temperature variation $(T-T_0)/T_0$ for small temperature changes

R is 0 for $t < t'$, that is

$$R(t) = 0, \quad \text{for} \quad t < 0 \quad \text{(causality)} \tag{2}$$

Summing up all contributions from $t' = -\infty$ to $t' = t$ gives

$$\begin{aligned} \gamma(t) &= \int_{-\infty}^{t} R(t-t')d\phi(t') = \int_{-\infty}^{t} R(t-t')\dot{\phi}(t')dt' \\ &= \int_{0}^{\infty} R(t'')\dot{\phi}(t-t'')dt''. \end{aligned} \tag{3}$$

Here $\dot{\phi}$ is the time derivative of ϕ. Introducing the composition $\circ$ as a shorthand notation for the linear operation "convolution with the time derivative" the equation (3) can be written

$$\gamma = R \circ \phi. \tag{4}$$

Notice, that the dimension of R is dim(output)/dim(input).

Classification of the response functions.

The name of the response function R is dependent on the type of input and output . γ and ϕ may be any of f, q, e, p; however they must belong to different causal classes. So if ϕ is e or p then γ has to be f or q and vice versa. This gives eight possible response functions, of which however two are identical with two other reducing the number to six:

With input of the *effort type:*

Compliance J, $q = J \circ e$,

Admittance Y, $f = Y \circ e$,

Lightness F, $f = F \circ p$.

With input of the *flow type:*

Modulus G, $e = G \circ q$,

Impedance Z, $e = Z \circ f$,

Inertance M, $p = M \circ f$.

One of the response functions is enough to characterise a system since the six functions are mathematically related. There are nevertheless several reasons for introducing a number of seemingly redundant functions. One is that experimentally it is convenient to be able to decide, which variable (input) is under control and which variable (output) is measured. In a physical theory of a phenomena it may also be different which r.f. it is most natural to consider. For certain of the simple network elements a particular response function is constant (independent of time). The scheme below gives an overview of the different response functions,

The time domain.

Let stimuli (input) be a sudden (step) change in ϕ from 0 to ϕ_0 at time $t = 0$. This can be expressed in terms of the Heaviside function

$$E(t) = \begin{cases} 0 & ,t < 0 \\ 1 & ,t > 0 \end{cases} \tag{5}$$

because then we have $\phi(t) = \phi_0 E(t)$. Differentiating the Heaviside function results in the deltafunction $\delta(t)$, which is 0 everywhere except for $t = 0$, at which point however it is infinite so that its integral is 1 Then we have

$$\gamma(t) = \int_0^\infty R(t')\phi_0\delta(t - t')dt' = \phi_0 R(t) \tag{6}$$

The response function $R(t)$ may thus be interpreted as the response $\gamma(t)$ to a unit step in stimuli at time $t = 0$, and it can be measured by a so-called step response experiment.

The frequency domain.

Equation (3) expresses, that the output is found by a convolution of the time derivative of the input with the response function. By a Fourier transformation convolution becomes into multiplication, differentiation into multiplication by $(i\omega)$ and integration into division by $i\omega$. This is why an analysis of a system behaviour is much easier in the frequency domain.

Thus consider a periodic input $\phi(t) = \phi_\omega e^{i\omega t}$. Since the equations are linear it is permissible to assume a complex input and extract the real part at the end of the calculation. We get

$$\begin{aligned} \gamma(t) &= \int_0^\infty R(t')(i\omega)\phi_\omega e^{i\omega(t-t')}dt' \\ &= (i\omega)\phi_\omega e^{i\omega t}\int_0^\infty R(t')e^{-i\omega t'}dt' \end{aligned} \tag{7}$$

The calculation shows that *a harmonic input results in a harmonic output.*. Writing $\gamma(t) = \gamma_\omega e^{i\omega t}$, we read off

$$\gamma_\omega = \hat{R}(\omega)\phi_\omega, \tag{8}$$

γ \ ϕ	q charge displacement	f current velocity	e voltage force	p magn. flux momentum
q charge displacement			G modulus stiffness	
f current velocity			Z impedance resistance	M inertance mass
e voltage force	J compliance creep function	Y admittance mobility		
p magn. flux momentum		F lightness		

Tabel 1 Table of the response functions, R and the possible relations, $\gamma = R \circ \phi$ between input, ϕ and output, γ. The operator $\circ$ is "convolution with the time derivative"in the time domain and plain multiplication in the frequency domain.

where

$$\hat{R}(\omega) = i\omega\mathcal{F}\{R, \omega\}, \tag{9}$$

and $\mathcal{F}\{R, \omega\}$ is the Fourier transformed of $R(t)$,

$$\mathcal{F}\{R, \omega\} = \int_{-\infty}^{\infty} R(t)e^{-i\omega t}dt. \tag{10}$$

Here the lower limit could be extended to $t = -\infty$, as usual in a Fourier transformation due to the causality condition (2). Notice that besides a Fourier transformation one has to multiply by $i\omega$. The *Frequency response function* $\hat{R}(\omega)$ becomes the function, that one would measure and define by (8) in a measurement with a harmonically varying input. $\hat{R}(\omega)$ is seen to have the same dimension as $R(t)$, and the limiting behaviour $\hat{R}(\omega)$ for $\omega \to 0$ or ∞ corresponds to $R(t)$ for $t \to \infty$ or 0 respectively. If the frequency response function is written in the modulus-argument form: $\hat{R}(\omega) = |\hat{R}(\omega)|e^{i\theta_\omega}$ then $\gamma(t) = |\hat{R}(\omega)|e^{i(\omega t+\theta_\omega)}$. Thus θ_ω is the number of radians that $\gamma(t)$ is phase shifted ahead of $\phi(t)$, whereas $|\hat{R}(\omega)|$ gives the amplitude ratio $|\gamma_\omega|/|\phi_\omega|$.

The return to $R(t)$ is performed by an inverse Fourier transformation

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{R}(\omega)}{i\omega}e^{i\omega t}d\omega \tag{11}$$

Laplace-Stieltjes transformation.

The Fourier transformation can be continued analytically for complex frequencies $\omega = \omega' + i\omega''$. In the lower half plane where $\omega'' < 0$ this can still be calculated by (10) since the factor $e^{\omega'' t}$ makes the integral converging for $t > 0$, while this factor makes no troubles for $t < 0$, since $R(t) = 0$ here. One often introduce the Laplace-frequency $s = i\omega$ and defines the s-frequency response function $\tilde{R}(s)$ by

$$\tilde{R}(s) = s\int_{0}^{\infty} R(t)e^{-st}dt = s\mathcal{L}\{R, s\} \tag{12}$$

that is s times the Laplace transformed of $R(t)$. The frequency response function can now be found (more generally) by

$$\hat{R}(\omega) = \lim_{s\to i\omega, s'>0} \tilde{R}(s) \tag{13}$$

Relations between the response functions.

I the frequency domain the relations between the response functions become very simple:
Within the two causal classes:

$$\tilde{M} = \frac{1}{s}\tilde{Z} = \frac{1}{s^2}\tilde{G} \quad , \quad \tilde{J} = \frac{1}{s}\tilde{Y} = \frac{1}{s^2}\tilde{F} \tag{14}$$

Between the two causal classes:

$$\tilde{J} = \frac{1}{\tilde{G}} \quad , \quad \tilde{Y} = \frac{1}{\tilde{Z}} \quad , \quad \tilde{F} = \frac{1}{\tilde{M}} \tag{15}$$

In the time domain it is only within the same causality class that it is fairly simple making a conversion (by integration or differentiation). Shift from one class to the other has to be done by solving an integral equation, which in fact would be easiest to solve by transforming to the frequency domain.

Compound systems.

It is readily found that connecting two systems by a Kirchhoff mesh, implying a common flow and an addition of the efforts results in a compound system that is additive in the (flow class) response functions (G, Z, M). On the other hand, connecting two systems by a Kirchhoff node, implying a common effort and an addition of the flows results in a compound system that is additive in the (effort class) response functions (J, Y, F). Using these rules combined with shift between the classes makes it swift to write up response functions of rather complex physical networks.