Streamline topologies of the periodic flow behind a cylinder

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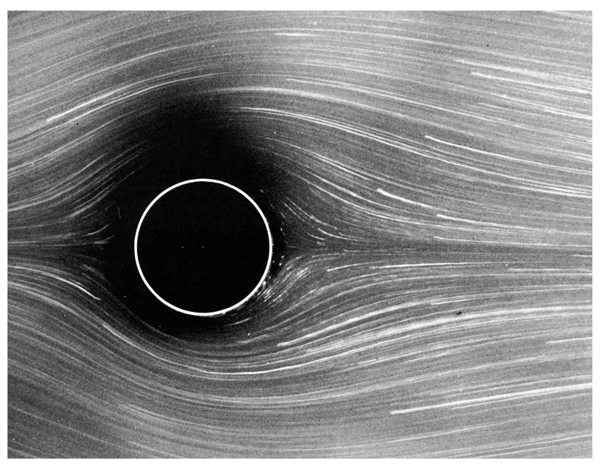
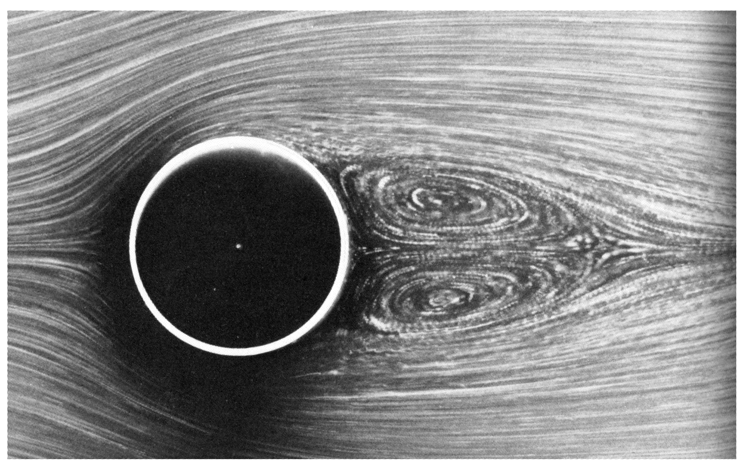
We present a topological fluid mechanical analysis of the streamline pattern of a two-dimensional flow behind a cylinder, including a comparison with recent numerical results.

Phenomenology

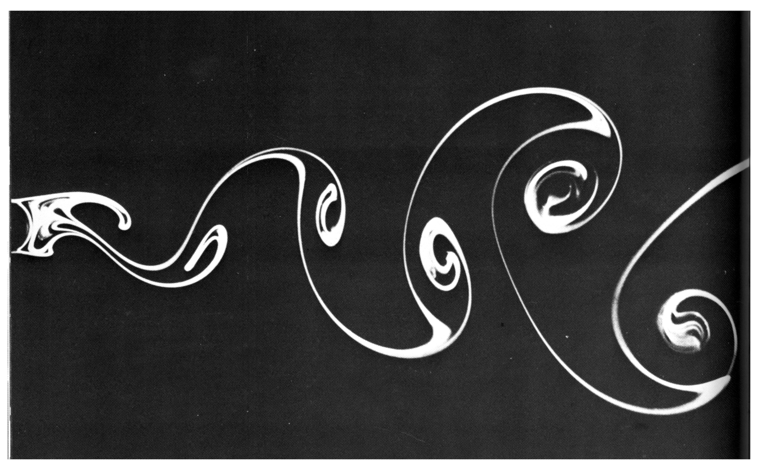
The two-dimensional flow behind a cylinder can be classified by different regimes depending on the Reynolds number.



Stationary creeping flow. Reynolds numbers up to approximately 7.



Stationary flow with two standing eddies (double eddy flow). Reynolds numbers approximately in the range 7 to 42.



Periodic flow with vortex shedding. Reynolds numbers approximately in the range 42 to 200.

Topological fluid mechanics

Given a time dependent velocity field, $\boldsymbol{u}(\boldsymbol{x}, t)$, an ordinary differential equation describing the instantaneous streamlines can be defined at each time, $t = t_0$:

$$\dot{\boldsymbol{x}} = \boldsymbol{u}(\boldsymbol{x}, t_0) \tag{1}$$

Topological fluid mechanics is the study of this type of equation using bifurcation theory of ordinary differential equations. Hereby information about the possible streamline topologies is obtained without solving the actual Navier-Stokes equations.

It is generally assumed that $\boldsymbol{u}$ fulfills the continuity equation for incompressible flow:

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2}$$

This leads to the existence of a stream function.

Topological model

The following assumptions lead to the simplest possible topological model of the structurally unstable streamline topology that exists in the transition from creeping flow to double eddy flow.

- the flow is two-dimensional
- no slip condition at the cylinder wall
- symmetry around a line perpendicular to the wall
- structural instability of the flow

The topological model is described by a stream function:

$$\psi(x, y) = xy^3 + x^3y^2 + O(6). \tag{3}$$

A versal unfolding of this stream function is constructed, and it is shown that three parameters are needed to capture all possible topologies.

By restricting the unfolding to symmetric flows, a bifurcation with the features of the transition from creeping flow to double eddy is found.

Interpretation of the transition to periodic flow

A transition from a stationary to a periodic flow, is in some cases, due to a Hopf bifurcation. By assuming the existence of a Hopf bifurcation it is shown that the sequence of topologies in the periodic flow immediately after the transition is described by a elliptical path, in the parameter space of the unfolding. The center of the ellipse corresponds to the transition flow.

If this result is applied to the special case of the double eddy flow, it is shown that the sequence of topologies is described by the inner curve on figure 1.

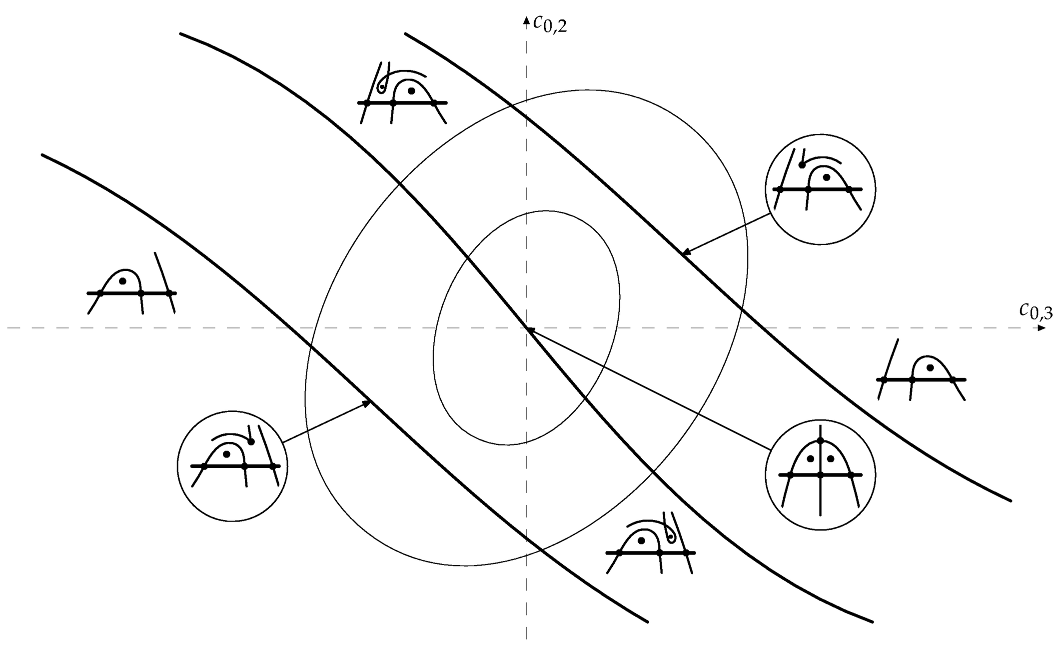
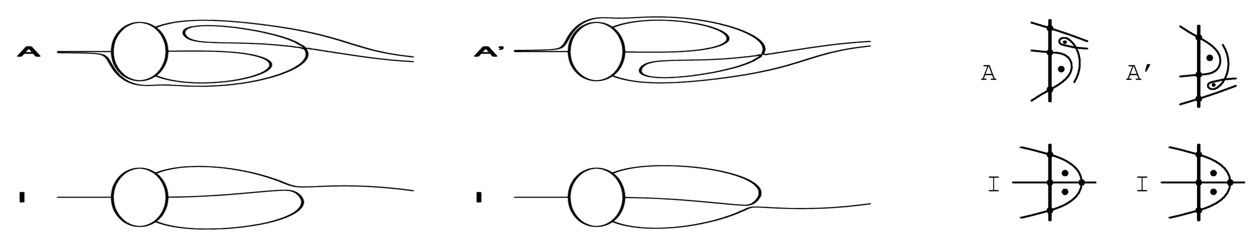


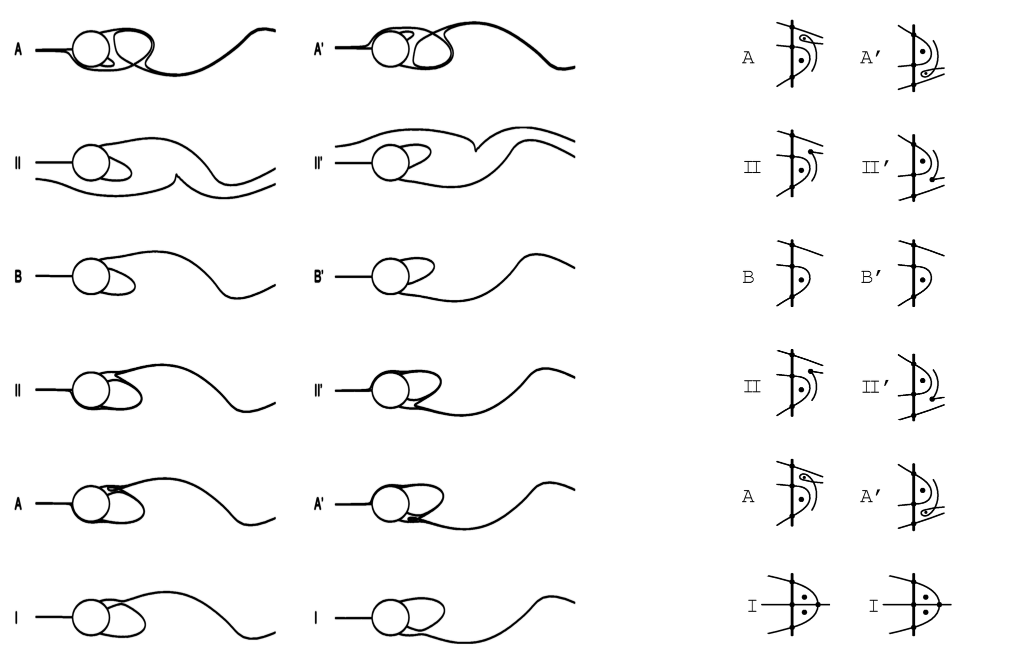
Figure 1: Part of a cross section from the three-dimensional parameter space. The parameters in the origin correspond to the double eddy flow.

Recent numerical results from R. Petersen [1], show that the actual sequence of topologies after the transition to periodic flow, is as illustrated in figure 2. By comparing this to the theoretical results illustrated in figure 1 perfect agreement is found.



Numerical results | Schematic interpretation

Figure 2: Sequence of topologies at $Re = 45$



Numerical results | Schematic interpretation

Figure 3: Sequence of topologies at $Re = 100$

The numerical results from R. Petersen [1], show that at $Re \simeq 46$ there is a transition to a periodic flow with a more complicated sequence of topologies (cf. figure 3). This sequence of topologies corresponds to the outer elliptical path in the unfolding in figure 1, thus indicating a high level of agreement between the topological model and the real flow.

[1] R. Petersen, *Dynamical Systems and Structures in the Flow Around a Cylinder*, Master Thesis No. 2002-01, Department of Mathematics, DTU