# Momentum and energy in an model with distributed sources

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**Abstract**

In the following paper we present the main results of a one term graduate student project[1]. The entire work is presented, in Danish, in Jakobsen & Niss [2000].

We analyse the form of the fluiddynamical equation of motion, in a two dimensional model with distributed sources and periodic boundary conditions. We conclude that the form of the equation of motion depends on the assumptions about the fluid entering and leaving the model. Futher we look at the total kinetic energy in the model, and how it depends on the assumptions.

## Introduction

The starting point of our work was Charles Peskin's two dimensional model of the flowpattern of the heart [Peskin, 1977]. The object of our work has been to determine which form the fluiddynamical equation of motion in the model should (or could) have. We start out by explaining why the form of the equation of motion has been an issue and proceed by presenting our results. We have worked theoretically with the form of the equation of motion and, subsequent to this, we have implemented an equation different from that originally used in the model, and studied the effects on the results of the model. In the

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following work we concentrate on the theoretical part of our work, as we consider our results here to be the most interesting. The heart model is not presented in this paper. We have made this choice, partly because the model, is thoroughly presented in several other papers (e.g. [Peskin, 1977]), and partly because the theoretical part of our results holds true for any two dimensional fluiddynamical model with distributed sources, sinks and periodic boundary conditions.

The central hypothesis of Peskin's modelling is that the blood can be modeled as an incompressible, homogeneous and Newtonian fluid[2]. The incompressibility is expressed in the equation of continuity,

$$\nabla \cdot \mathbf{u} = 0$$

which, along with the assumptions that the fluid is Newtonian and homogeneous, leads to the following (standard) equation of motion, the Navier-Stokes equation,

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p - \mu \nabla^2 \mathbf{u} + \mathbf{F} \tag{2}$$

Where $\mathbf{u}$ and $t$ represent velocity and time. $F$ is force per unit area, which is analogous to a volume force in three dimensions. $p$ represents pressure and has the dimension force per unit length. $\rho$ and $\mu$ are the (two dimensional) density and viscosity. Spatial and time dependence should be understood.

One of the main charecteristics, of Peskin's model is that it has periodic boundary conditions. The consequence of the periodic boundary conditions is that fluid cannot enter or leave the modeled area at the boundary. In its early use this did not cause any problems because only a limited part of the heart cycle was represented in the model. However as the model advanced to modelling a full or even a couple of heartbeats, it soon became necessary to have fluid leaving and entering the model. Peskin solved this problem by placing distributed sources and sinks in the model. The distribution of source and sinks are given by a function ($\psi$) that depends on time and space. $\psi$ gives the area of fluid[3] entering per area of the model per second, thus it has the dimension "per second".

The mathematical consequence of the introduction of $\psi$ is that the equation of continuity is modified to,

[2]The assumption that the fluid is Newtonian leads to the following expression for the stress tensor

$$T_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \nabla \cdot \mathbf{u}\delta_{ij} \right) \tag{1}$$

[3]The model is 2-dimensional, hence area.

$$\nabla \cdot \mathbf{u} = \psi \tag{3}$$

Thus the above equation has been a part of the model since the introduction of distributed sources[4] in the model. Apparently the form of the equation of motion was not reconsidered when the sources were introduced, although the equation of continuity plays a crucial part in the derivation of the equation of motion. This led to the concern that the model contained inconsistencies, and this is why the form of the equation of motion is worth discussing. We will continue by doing so.

## The effect of the sources on the momentum

The resulting form of the equation of motion in a system with distributed sources, depends on the assumptions made about the momentum of the fluid that leaves or enters the model. It is obvious that it is desirable that the fluid leaving the model takes its momentum with it as it leaves. It is less obvious how much momentum should be designated to fluid entering the model.

Since the fluid is assumed incompressible and homogeneous, momentum is proportional to the velocity of the fluid, thus momentum and velocity can be used equivalently.

We have considered two possible assumptions about the momentum of the fluid entering the model. The first being that the fluid enters with no velocity (this assumption has been suggested by Peskin in an unpublished note [Peskin, 1998]), and the second being that it enters with the velocity the fluid in the model has at the source.

Both assumptions are combined with equation 3, and the assumption that the fluid, when leaving the model, takes it momentum with it. The equation of motion has been derived by looking at the change of momentum in a randomly chosen fixed area ($A$), contained in a closed curve ($l$).

The assumption that fluid entering the model has no momentum leads to the following equation regarding the $i$'th component of the momentum ($\mathbf{P}$),

$$\frac{dP_i}{dt} = \int_A \frac{\partial u_i \rho}{\partial t}\, dA = \int_l T_{ij} n_j \, dl + \int_A F_i dA - \int_l u_i \rho \, u_j n_j \, dl + \int_A u_i \rho \psi^- \, dA \tag{4}$$

Where $i, j \in x, y$ and we use Einstein's summation convention. $T_{i,j}$ is the stress tensor, and $\psi^- = \psi$ if $\psi < 0$ else $\psi^- = 0$.

[4]In the rest of this text we use "sources" as an abbreviation for both sources and sinks.

By using Gauss' theorem, the continuity of the integrand, and the fact that the area of integration is randomly chosen we get the following equation of motion after some manipulations,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=-\nabla p-\mu\nabla^2\mathbf{u}+\mathbf{F}-\psi^+\mathbf{u} \tag{5}$$

where $\psi^+ = \psi - \psi^-$.

Thus, we have an equation of motion which is modified, in comparison to equation 2 due to the distributed sources in the model.

The assumption that fluid entering the model has the same velocity as the fluid in the model at the point where it enters, leads to the following equation,

$$\frac{dP_i}{dt}=\int_A \frac{\partial u_i\rho}{\partial t}\,dA=\int_l T_{ij}n_j\,dl+\int_A F_i dA-\int_l u_i\rho\,u_jn_j\,dl+\int_A u_i\rho\psi\,dA \tag{6}$$

which leads to

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=-\nabla p-\mu\nabla^2\mathbf{u}+\mathbf{F}$$

This is the same equation as equation 2. Thus an equation of motion equivelant to the ordinary equation of motion for incompressible fluids is found, although the equation of continuity does not have its standard form. This is especially interesting because this is the equation of motion which in fact has been used in Peskin's model.

Thus we have shown that the use of the standard Navier-Stokes equation along with equation 3 does not lead to any inconsistencies, but that it is the consequence of sensible assumptions about the momentum at the sources. We have shown that different assumptions lead to different equations of motion, therefore it is important to discuss which assumptions are the most appropriate for the model in question.

We have not studiet which of the two assumptions is the more theoretically correct (from a physiological point of view) in the case of the heart. We believe that both assumptions are plausible, but other assumptions may be plausible as well. Evidently it is possible to choose the momentum of the entering fluid arbitrarily. However in the case of the heart it is not reasonable to choose a velocity greater than the current velocity at the source, since blood enters the heart at a low speed and is accelerated by the work of the heart muscle.

We have implemented equation 5 in the computer program that runs the heart model, and have found that this change only makes very minor differences to the velocity field. This is due to the fact that the velocities of the fluid at the sources where fluid enters the model are small. In other models that might not be the case, and the differences in the velocity fields could be crucial.

## The effect of the sources on the total kinetic energy

In our work with the model we needed a way to characterise the total state of the system. We have calculated the change in total kinetic energy of the system as a function of time. This calculation is based on a method from Peskin [1998].

By rearranging the terms of equation 5 we get,

$$\rho \frac{\partial u_i}{\partial t} = -\rho(\mathbf{u} \cdot \nabla) u_i + F_i + \mu \nabla^2 u_i - \frac{\partial p}{\partial x_i} - u_i \rho \psi^+ \tag{7}$$

multiplying by $u_i$, summerising over $i$ and integrating over the computation domain we get the following after some heavy reduction,

$$\begin{aligned} \int_\Omega \rho u_i \frac{\partial u_i}{\partial t}\, dA = \\ \int_\Omega u_i F_i \, dA - \int_\Omega \mu \left(\frac{\partial u_i}{\partial x_j}\right)^2 dA + \int_\Omega p\psi \, dA - \int_\Omega \frac{1}{2} u_i u_i \rho |\psi| \, dA \end{aligned} \tag{8}$$

In the calculation the periodic boundary conditions are exploited together with the fact that the domain is rectangular.

It is seen that the left side reduces to,

$$\int_\Omega \rho u_i \frac{\partial u_i}{\partial t}\, dA = \int_\Omega \frac{1}{2} \rho \frac{\partial |\mathbf{u}|^2}{\partial t}\, dA$$

which exactly is the rate of change of total kinetic energy.

An equivalent calculation based on equation 2 gives,

$$\begin{aligned} \int_\Omega \rho u_i \frac{\partial u_i}{\partial t}\, dA = \\ \int_\Omega u_i F_i \, dA - \int_\Omega \mu \left(\frac{\partial u_i}{\partial x_j}\right)^2 dA + \int_\Omega p\psi \, dA + \int_\Omega \frac{1}{2} u_i u_i \rho \psi \, dA \end{aligned} \tag{9}$$

It is seen that the two equations for the rate of change of the total kinetic energy only differ in the last term, and do so in a way that can be interpreted. In both cases the liquid leaving the model $(\psi < 0)$ results in a loss of energy. This corresponds well to the fact that the liquid takes along its momentum when leaving the model.

In equation 8 the liquid entering the model also results in a loss of energy. This is because the liquid in the model has to accelerate the liquid entering, and this process results in an energy loss. As Peskin points out [Peskin, 1998], this process is analogous to an inelastic collision.

In equation 9, the liquid entering the model provides a positive amount of energy, which corresponds well to the assumption that the liquid entering the model has a positive velocity.

We have implementet a discrete version of the energi calculations in the program which runs the model. In figure 1 the total kinetic energy is plottet as a function of time. Data from the original model[5] and the modified version of the model[6] is shown along with the difference between the two. The figure shows only minor differences between the to versions of the model, and the same result is, as earlier mentioned, obtained by studying the velocety fields of the two versions of the model.

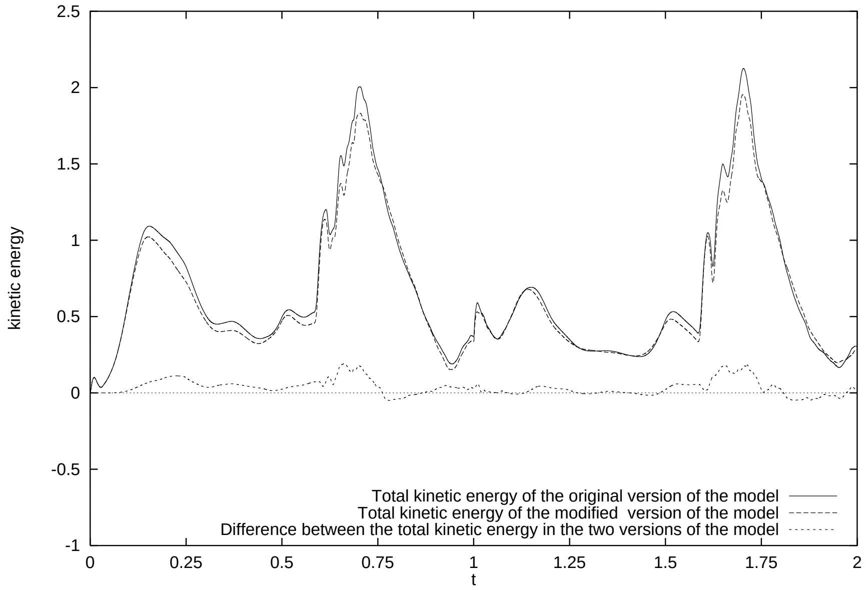


**Figure 1** Total kinetic energy (devided by density) as a funktion of time. Time of a heart beat and length of the domain of calculation are used as units of time and length.

# Conclusion

We conclude that it is important to decide what the assumptions about the momentum of the liquid entering and leaving the model should be, because different assumptions lead to different equations of motion. In some models this difference might be of great importance, but in the model we have worked with the differences in the results of the model are very minor.

[5]The original model corresponds to equation 2 and 9.

[6]The modified version of the model corresponds to equation 5 and 8

We have also shown that a special set of assumptions leads to the standard Navier-Stokes equation. This means that current models using this equation and having the described properties (distributed sources and periodic boundary conditions), do in fact handle the momentum in a consistent way, if the described assumptions can be applied.

## References

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