Additional note for: Exercise in thermal analysis

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This note tries to clarify a few things which are treated in the course, especially with respect to "linear and non-linear" response to a periodic input.

You should study the main note in detail (and try to do the exercises in that) before reading this.

Amplitudes

This section is meant to elaborate a bit on how to understand the complex amplitudes in the Fourier expansion of an periodic signal.

A periodic signal $A(t)$ can generally be written as a sum of harmonic terms:

$$A = A_0 + |A_1|\cos(\omega t + \phi_1) + \ldots + |A_n|\cos(n\omega t + \phi_n) \tag{1}$$

where A_0, $|A_k|$ are real amplitudes and ϕ_k phases.

Such a sum of harmonic terms can be written as

$$A = \frac{1}{2}\left(A_0 + |A_1|e^{i(\omega t+\phi_1)} + |A_2|e^{i(2\omega t+\phi_2)} \ldots + |A_n|e^{i(n\omega t+\phi_n)} + c.c\right). \tag{2}$$

where $+c.c.$ means that the complex conjugated of all terms in the parenthesis are to be added (including the conjugated of the real constant).

By introducing the complex amplitude $A_k = |A_k|e^{i\phi_k}$ and a shorthand notation $E_k = e^{ik\omega t}$ the sum can be written as

$$A = \frac{1}{2}(A_0 + A_1E_1 + A_2E_2 \ldots + A_nE_n + c.c.), \tag{3}$$

.

A_1 is what we normally understand as the complex amplitude (and which you spend time on in the "Physical modeling course"), namely the complex amplitude of the fundamental frequency ω. You may also have come about A_0 which is the "DC" amplitude, that is the amplitude of the not oscillating part of the signal.

The higher harmonic amplitudes are in no way different, they tell the amplitude of the part which vary with higher frequencies.

Why can $I(t)$, $P(t)$,$\Delta T(t)$ be expanded

Lets assume that the generator provides a perfect single harmonic output, that is

$$\begin{aligned} U(t) &= \frac{1}{2}\left(U_1 E_1 + U_1^* E_1^*\right) && (4)\\ &= \frac{1}{2} U_1 \left(E_1 + E_1^*\right) && (5)\\ &= U_1 \cos(\omega t) && (6) \end{aligned}$$

(u_1 is a real number if the phase is 0).

Ohm's law holds at all times, and from that followed the basic equation for the measured voltage

$$V(t) = \frac{R_{\text{pre}}}{R_{\text{pre}} + R(T(t))} U(t). \tag{7}$$

From this it does *not* follow that $V(t)$ also is periodic (or has the same period time), this depends in $R(t)$. However R is controlled by the same harmonic input so it is intuitively reasonable to assume that this is also varying in a harmonic way with the same (or shorter) period time [1] (this is for sure the case if it is possible to write up R as a Taylor expansion of U). $V(t)$ will therefore[2] also be periodic with the same period time and can therefor be written as

$$\begin{aligned} V(t) &= \frac{1}{2}\left(V_1 E_1 + V_2 E_2 + \ldots + c.c.\right) && (8)\\ & && (9) \end{aligned}$$

From this it follows that the thermal power and the temperature (difference to the cryostat) also can be written as

$$\begin{aligned} P(t) &= \frac{1}{2}\left(P_1 E_1 + P_2 E_2 + \ldots + c.c.\right) && (10)\\ \Delta T(t) &= \frac{1}{2}\left(T_1 E_1 + T_2 E_2 + \ldots + c.c.\right) && (11) \end{aligned}$$

[1] Examples exists of "passive" systems that has a period which is twice the input period.

[2] if you are troubled by the handwaving arguments, this can be tested by checking that the amplitudes of V ($V_1, V_2, \ldots$) are time independent.

The important thing to remember when using these "Fourier" expansions is that it is unique. That is if you have some function/equality with such expansion on both sides, the individual coefficient have to be the same.

Interpretation

For the later analysis the two important quantities are P and ΔT so lets look a bit more on the interpretation of the amplitudes.

P_0, T_0 the amplitudes of the DC (time independent) thermal power (current) and temperature

P_1, T_1 (Complex) amplitude of oscillations with 1ω as frequency

P_2, T_2 (Complex) amplitude of oscillations with 2ω as frequency (This is the dominant term).

When thinking of the full periodic power signal, think of it as consisting of an sum of harmonics which can be treated on individual basis.

Thermal impedance

When considering the thermal (complex) impedance Z_{th} we are "back" to traditional linear response theory. We assume that the current (heat current), P, and voltage (temperature), ΔT, are described by harmonic functions at a given frequency ω_{th}, that is;

$$P = \tilde{P}_{\omega_{th}} e^{i\omega_{th} t} \tag{12}$$
$$\Delta T = \tilde{T}_{\omega_{th}} e^{i\omega_{th} t} \tag{13}$$

(where I have just written the complex voltage and current, but where one should remember that the real time-dependent signal is found by taking the real part).

The thermal impedance (at the frequency ω_{th}) is then defined as

$$\tilde{Z}_{th}(\omega_{th}) = \frac{\tilde{T}_{\omega_{th}}}{\tilde{P}_{\omega_{th}}} \tag{14}$$

The problem is now to realize the connection between the amplitudes and frequencies from the two treatments.

In the exercise we find expressions for P_2 and T_2 from equation 10 and 11. But what they express is exactly the amplitudes in the sense of equation 12 and 13 for a thermal signal varying at frequency $\omega_{th} = 2\omega$. That is

$$\frac{T_2}{P_2} \tag{15}$$

is the thermal impedance evaluated at a frequency of 2ω.

So if we make a frequency scan, we generally find

$$Z_{th}(\omega_{th}) = \frac{T_2(\omega)}{P_2(\omega)} \tag{16}$$

where $\omega_{th} = 2\omega$.