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Newton's discrete dynamics

Søren Toxvaerd[a]

Department of Science and Environment, Roskilde University, Postbox 260, 4000 Roskilde, Denmark

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Abstract In 1687, Isaac Newton published PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA, where the classical analytic dynamics was formulated. But Newton also formulated a discrete dynamics, which is the central difference algorithm, known as the Verlet algorithm. In fact, Newton used the central difference to derive his second law. The central difference algorithm is used in computer simulations, where almost all Molecular Dynamics simulations are performed with the Verlet algorithm or other reformulations of the central difference algorithm. Here, we show that the discrete dynamics obtained by Newton's algorithm for Kepler's equation has the same solutions as the analytic dynamics. The discrete positions of a celestial body are located on an ellipse, which is the exact solution for a shadow Hamiltonian nearby the Hamiltonian for the analytic solution.

1 Introduction

In 1687, Isaac Newton published PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA ($Principia$) [1] with the foundation of the classical analytic dynamics. Newton described the dynamics of an object by means of a differential equation, and in the Lagrange–Hamilton formulation of the classical dynamics, the position $\mathbf{r}(t)$ and momentum $\mathbf{p}(t)$ are analytic dynamical variables of a coherent time. But in $Principia$, Newton also derived a discrete dynamics, where a celestial body's positions are obtained at discrete times. The discrete Newtonian dynamics has the same invariances as the analytic dynamics, but differs fundamentally by that only the discrete positions are dynamic variables of the discrete time.

Today almost all numerical integrations of classical dynamics are performed by Newton's discrete dynamics, by updating the positions at discrete times. The Newtonian dynamics is the classical limit dynamics of the relativistic quantum dynamics, and the fundamental length and time in quantum dynamics are the Planck length $l_{\mathrm{P}} \approx 1.6 \times 10^{-35}$m and Planck time $t_{\mathrm{P}} \approx 5.4 \times 10^{-44}$ s [2]. They are immensely smaller than the differences in step lengths and the time increments used in the numerical integration by discrete dynamics, so the difference between the two dynamics in the classical limit for the dynamics of heavy objects with slow motions is $nihil$.

Newton's discrete dynamics has the same qualitative behaviour as the analytic. It is time reversible, symplectic [3] and has the same invariances as the analytic dynamics: conservation of momentum, angular momentum and energy [4]. It is furthermore possible by an asymptotic

[a] e-mail: st@ruc.dk (corresponding author)

expansion to make it probably, that the positions of an object obtained by Newton's discrete dynamics are located on the analytic trajectory for a *shadow Hamiltonian* nearby the Hamiltonian for the corresponding analytic dynamics [5]. If that is the case the numerical generation of positions in computer simulations (Molecular Dynamics) is the exact positions for the discrete dynamics obtained by Newton's central difference algorithm. Here, we show that the dynamics, obtained by solving Kepler's equation for celestial objects by discrete dynamics, give stable orbits which only differ marginally from the corresponding analytic orbits and with a strong indication of a shadow Hamiltonian for the dynamics.

2 Newton's discrete dynamics: the central difference algorithm

Newton's second law relates an object with mass m at the position, **r**(t), momentum, **p**(t), at time t with the force **F(r)**. The English translation [6] of the Latin formulation of Newton's second law is

The alteration of motion(momentum) *is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*, i.e.,

$$\mathbf{F}(r) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \tag{1}$$

and in *Section II*, Newton derived an interesting relation:

PROPOSITION I. THEOREM I. The areas, which resolving bodies describe by radii drawn to an immovable centre of force do lie in the same immovable planes and are proportional to the times in which they are described.

Newton noticed, that (see Fig. 1): *For suppose the time to be divided into equal parts, and in the first part of that time let the body by its innate force describe the right line AB. In the second part of that time, the same would (by Law I.), if not hindered, proceed directly to c, along the line Bc equal to AB; so that by the radii AS, BS, cS, drawn to the centre, equal areas ASB, BSc, would be described. But when the body is arrived at B, suppose that a centripetal force acts at once with a great impulse, and turning aside the body from the right line Bc, compels it afterwards to continue its motion along the right line BC. Draw cC parallel to BS meeting BC in C; and at the end of the second part of the time, the body (by Cor. I. of the Laws) will be found in C, in the same plane with triangle ASB Join SC, and because SB and Cc are parallel, the triangle SBC will be equal to the triangle SBc, and therefore also to the triangle SAB.*

So according to Newton's $PROPOSITION$, the particle moves with constant momentum, $m(\mathbf{r}_B(t_0+\delta t) - \mathbf{r}_A(t_0))/\delta t$ from the position $\mathbf{r}_A(t_0)$ to the position $\mathbf{r}_B(t_0+\delta t)$ in the time interval $t \in [t_0, t_0+\delta t]$, where a force, $\mathbf{F}(\mathbf{r}_B)$ instantaneously changes the momentum. This formulation of the discrete updating of positions: $\mathbf{r}_A(t_0)$, $\mathbf{r}_B(t_0+\delta t)$ $\mathbf{r}_C(t_0+2\delta t)$,.. with constant momentum in the time intervals between the updating is *the central difference algorithm*

$$m\frac{\mathbf{r}(t_n+\delta t) - \mathbf{r}(t_n)}{\delta t} = m\frac{\mathbf{r}(t_n) - \mathbf{r}(t_n-\delta t)}{\delta t} + \delta t\mathbf{F}(t_n). \tag{2}$$

The algorithm determines the $n+1$'the position from the two previous positions by

$$\mathbf{r}(t_n+\delta t) = 2\mathbf{r}(t_n) - \mathbf{r}(t_n-\delta t) + \frac{\delta t^2}{m}\mathbf{F}(t_n), \tag{3}$$

and this formulation of Newton's central difference algorithm is the so called Verlet algorithm [7,8], which is used in Molecular Dynamics simulations [9–11]. The algorithm can be reformulated, if one updates the positions in two steps with $\mathbf{v}(t_n+\delta t/2) \equiv (\mathbf{r}(t_n+\delta t) - \mathbf{r}(t_n))/\delta t$:

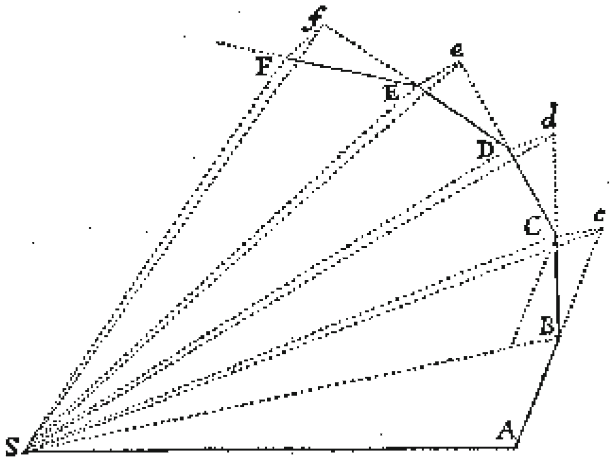

Fig. 1 Newton's figure in Principia, at his formulations of the discrete dynamics. The discrete positions are A: $\mathbf{r}_A(t_0)$; B: $\mathbf{r}_B(t_0 + \delta t)$; C: $\mathbf{r}_C(t_0 + 2\delta t)$, etc. The deviation from the straight line ABc (Newton's first law) to point C is caused by a force at S, acting on the object at point B at time $t_0 + \delta t$

$$\begin{aligned} \mathbf{v}(t_n + \delta t/2) &= \mathbf{v}(t_n - \delta t/2) + \frac{\delta t}{m}\mathbf{F}(t_n) \\ \mathbf{r}(t_n + \delta t) &= \mathbf{r}(t_n) + \delta t \mathbf{v}(t_n + \delta t/2), \end{aligned} \tag{4}$$

and this reformulation is named the "leap-frog" algorithm. It is the discrete version of Euler's equations for Newton's analytic dynamics [12].

There are several things to note about Newton's formulation of the discrete dynamics. According to Newton, the force acts *at once with a great impulse*, i.e., the forces are discrete, it acts only at the discrete times t_n and the object is not exposed to the force within the time intervals between the discrete times where it moves with constant momentum as Newton explicit notes: *(by Law I.)*.

Another thing to note is, that Newton in *Principia* did not write that the constant areal of the triangles is Kepler's second law for the planets orbits around the Sun. But Newton must have noticed this fact and must have realized that his dynamics, even in the discrete version, most likely could explain the celestial dynamics. The equal area of the triangles and Kepler's second law is according to the proof in *PROPOSITION I. THEOREM I.* valid for any central force between two celestial objects. It is a consequence of the conserved angular momentum in the discrete and analytic dynamics (see later). The *Principia* is written long time after Newton in fact had formulated his classical dynamics, and Newton solved Kepler's equation (geometrically!) for the analytic dynamics in *Principia*.

A third thing to note is the continuation of *PROPOSITION I. THEOREM I.*: *Now let the number of those triangles be augmented, and their breadth diminished in infinitum; and (by Cor.4, Lem, III) their ultimate perimeter ADF will be a curve line: and therefore the centripetal force, by which the body is perpetually drawn back from the tangent of this curve, will act continually; and any described areas SADS, SAFS, which are always proportional to the times of description, will, in this case also, be proportional to those times. Q.E.D.*. So Newton used the central difference to obtain his analytic dynamics and noticed, that by letting the time increment go to zero he obtained not only a curve line and a continuous force, but also maintained the constant area of the triangles. But he did not mentioned Kepler's second law.

There exist several other reformulations of the central difference algorithm [10,11]. The Verlet algorithm was derived by L. Verlet by a forward and backward Taylor expansion, and the algorithm and its many reformulations are normally presented as a third order predictor of the positions, obtained by Taylor expansions. Newton was well aware of Taylor expansions, but he did not use it to formulate a discrete dynamics. It is the other way around, Newton used the discrete dynamics to obtain the analytic dynamics and his second law.

Before the formulation of the discrete dynamics for a celestial body is presented, the solution of Kepler's equation for analytic dynamics is summarized in the next section.

3 The solution of Kepler's equation

3.1 The analytic solution of Kepler's equation

Newton solved in *Principia*, Kepler's equation for the orbit of a planet. The solution of Kepler's equation [13]

$$\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = -\frac{gMm}{r(t)^2}\hat{\mathbf{r}} \tag{5}$$

for a planet with the gravitational constant g and mass m at the position $\mathbf{r}(t)$ from the Sun at the origin [14] with mass M relates the constant energy,

$$E = 1/2m\mathbf{v}(t)\cdot\mathbf{v}(t) - gMm/r(t), \tag{6}$$

with the semi-major axis in an ellipse

$$a = -gMm/2E. \tag{7}$$

The longest distance $r_{\max}$ (aphelion) from the Sun is

$$r_{\max} = 2a - r_p, \tag{8}$$

where r_p is the shortest distance (perihelion) to the Sun. The eccentricity, ϵ, is

$$\epsilon = \frac{r_{\max} - r_p}{r_{\max} + r_p} = 1 - \frac{r_p}{a}, \tag{9}$$

and the semi minor axis, b, is

$$b = a\sqrt{1-\epsilon^2}. \tag{10}$$

With the major axis in the x-direction the planet moves in a stable elliptic orbit

$$\frac{(x(t) - (a - r_p))^2}{a^2} + \frac{y(t)^2}{b^2} = 1, \tag{11}$$

for

$$0 \leq \epsilon < 1, \tag{12}$$

within a orbit period

$$T(orbit) = 2\pi\sqrt{a^3/gM}. \tag{13}$$

The velocity at perihelion, $\mathbf{v}_p(t)) = (0, vy_p)$, is in the y-direction, and the energy is

$$E = 1/2mvy_p^2 - gMm/r_p, \tag{14}$$

and since $1/a = -2E/gMm = -mvy_p(t)^2 + 2/r_p$, the limit values for elliptic orbits can be expressed by the maximum velocity as

$$\sqrt{gM/r_p} \leq vy_p < \sqrt{2gM/r_p}. \tag{15}$$

Let the planet at time $t_0 = 0$ be in the perihelion of the elliptic orbit with the maximum velocity $\mathbf{v}_p = (0, vy_p)$ at the shortest distance, $\mathbf{r}_{\min} = (x(t_0), y(t_0)) = (-r_p, 0)$, from the Sun, which is located at the origin. The classical orbit of a planet can be obtained from these four parameters: gM, m, r_p, vy_p (or: $gM, m, r_{\max}, vy_{\min}$ at aphelion).

3.2 Kepler's orbit obtained by Newton's central difference algorithm

The discrete dynamics can be obtained from the same parameters, gM, m, r_p, vy_p together with the discrete time increment δt. Newton's discrete dynamics for the $n+1$'th change of position of a planet is

$$\frac{\mathbf{r}(t_{n+1}) - \mathbf{r}(t_n)}{\delta t} = \frac{\mathbf{r}(t_n) - \mathbf{r}(t_{n-1})}{\delta t} - \frac{gMm\delta t}{r(t_n)^2}\hat{\mathbf{r}}(t_n). \tag{16}$$

An important fact is that the algorithm relates a new position with the two previous positions and the forces at the time, where the forces act, i.e., the momentum (or velocity) is not a dynamical variable in the discrete dynamics, and any expression for velocity, and thereby the kinetic energy is ad hoc.

The discrete time evolution with the constant time increment δt, obtained by Newton's central difference algorithm, starts from either two sets of positions, $\mathbf{r}(t_0), \mathbf{r}(t_0 - \delta t)$ (Verlet algorithm), or, as Newton illustrated, from a position $\mathbf{r}(t_0)$ and a previous change of position $\mathbf{r}(t_0) - \mathbf{r}(t_0 - \delta t) \equiv \delta t \mathbf{v}(t_0 - \delta t/2)$, in the time interval $t \in [t_0 - \delta t, t_0]$ (Leap frog or implicit Euler algorithm). The velocity $\mathbf{v}(t_n)$ at the time where the force acts, at the position $\mathbf{r}(t_n)$, is in general obtained by a central difference

$$\mathbf{v}(t_n) = \frac{\mathbf{v}(t_n + \delta t/2) + \mathbf{v}(t_n - \delta t/2)}{2} = \frac{\mathbf{r}(t_n + \delta t) - \mathbf{r}(t_n - \delta t)}{2\delta t}. \tag{17}$$

Newton's discrete time reversible dynamics has the same three invariances as his analytic dynamics. It conserves the (total) angular momentum, $\mathbf{L}$. The angular momentum, $\mathbf{L}(t_n)$, for a planet at the n'th time step (and using the Verlet formulation, Eq. (17) and the fact, that the force is in the direction of the discrete position) is

$$\begin{aligned}\frac{2\delta t}{m}\mathbf{L}(t_n) &= \mathbf{r}(t_n) \times (\mathbf{r}(t_{n+1}) - \mathbf{r}(t_{n-1})) \\ &= \mathbf{r}(t_n) \times (2\mathbf{r}(t_n) - 2\mathbf{r}(t_{n-1})) \\ &= \mathbf{r}(t_n - 1) \times (\mathbf{r}(t_n) + \mathbf{r}(t_n)) = \\ &= \mathbf{r}(t_n - 1) \times (\mathbf{r}(t_n) - \mathbf{r}(t_{n-2})) = \frac{2\delta t}{m}\mathbf{L}(t_{n-1}).\end{aligned} \tag{18}$$

It is straightforward to prove that the constant area of the triangles in Newton's formulation of the discrete dynamics (Fig. 1) is a consequence of the conserved angular momentum.

If one determines the energy at the n'th time step by

$$E_{\text{disc}}(t_n) = \frac{1}{2}m\mathbf{v}(t_n)^2 - \frac{gMm}{r(t_n)}, \tag{19}$$

it fluctuates during the discrete time propagation, although the mean value remains constant.

3.3 The shadow Hamiltonian for the central difference algorithm

The points obtained by Newton's central difference algorithm for a simple harmonic force are located on a harmonic trajectory of a harmonic "shadow Hamiltonian" $\tilde{H}(\mathbf{q},\mathbf{p})$ [5], with position $\mathbf{q}$ and momentum $\mathbf{p}$ in the Lagrange–Hamilton equations. The shadow Hamiltonian $\tilde{H}$ for a symplectic and time-reversible discrete algorithm can in general be obtained from the corresponding $H(\mathbf{q},\mathbf{p})$ for the analytic dynamics by an asymptotic expansion in the time increment δt, if the potential energy is analytic [15–17],

$$\tilde{H} = H + \frac{\delta t^2}{2!} g(\mathbf{q},\mathbf{p}) + \mathcal{O}(\delta t^4). \tag{20}$$

The corresponding energy invariance, $\tilde{E}$, for the discrete dynamics in Cartesian coordinates for N particles is [5,18,19]

$$\begin{aligned}\tilde{E}_n = U(\mathbf{R}_n) + \frac{1}{2} m\mathbf{V}_n^2 + \frac{\delta t^2}{12}\mathbf{V}_n^T \mathbf{J}(\mathbf{R}_n)\mathbf{V}_n \\ -\frac{\delta t^2}{24m}\mathbf{F}_n(\mathbf{R}_n)^2 + \mathcal{O}(\delta t^4),\end{aligned} \tag{21}$$

where $\mathbf{J}$ is the Hessian, $\partial^2 U(\mathbf{q})/\partial \mathbf{q}^2$, of the potential energy function $U(\mathbf{q})$, the velocity of the N particles is $\mathbf{V}_n \equiv (\mathbf{v}_1, \ldots, \mathbf{v}_N)$, and the force with position $\mathbf{R} \equiv (\mathbf{r}_1, \ldots, \mathbf{r}_N)$ is $\mathbf{F}(\mathbf{R}) \equiv (\mathbf{f}_1(\mathbf{R}), \ldots, \mathbf{f}_N(\mathbf{R}))$.

The observed energy fluctuations for a complex system decrease by a factor of hundred or even more by including these terms in the expression for the energy and it indicates that the expansion is rapidly converging for relevant time increments [19,20].

The shadow energy at the n'th step for a planet, attracted by the Sun at a fixed position at the origin, can be obtained from the expressions in Appendix A in [19]. It is

$$\begin{aligned}\tilde{E}(t_n) = E_{\text{disc}}(t_n) - \frac{\delta t^2}{12}\Bigg(\frac{3gMm}{r(t_n)^5}(\mathbf{v}(t_n)\mathbf{r}(t_n))^2 \\ + \frac{gMm}{r(t_n)^3}\mathbf{v}(t_n)^2\Bigg) - \frac{\delta t^2 (gMm)^2}{24 r(t_n)^4} + \mathcal{O}(\delta t^4).\end{aligned} \tag{22}$$

4 The orbit of a planet obtained by Newton's discrete algorithm

The positions of a planet are obtained by Newton's central difference algorithm. The positions are determined by the time increment δt and by the same parameters as the analytic curve, e.g., gM, m, r_p and vy_p. The curves through the points are almost identical to the analytic ellipses, and the discrete dynamics obeys the same condition for a stable elliptic orbit as the analytic dynamics (Eq. (15)). Figure 2 shows the orbits, obtained with different start values of the velocity, vy_p [21].

The generation of positions by the central difference algorithm needs either two consecutive start positions, $\mathbf{r}(t_0 - \delta t)$ and $\mathbf{r}(t_0)$, or $\mathbf{r}(t_0)$ and $\mathbf{v}(t_0 - \delta t/2)$. It is convenient to start the dynamics in perihelion (or aphelion) where $vx(t_0) = 0$. Due to the time reversibility of the discrete dynamics $vy(t_0 + \delta t/2) = vy(t_0 - \delta t/2)$ and $vx(t_0 + \delta t/2) = -vx(t_0 - \delta t/2)$ at perihelion. The first discrete position away from the perihelion, $x(t_0 + \delta t)$, $y(t_0 + \delta t)$, is

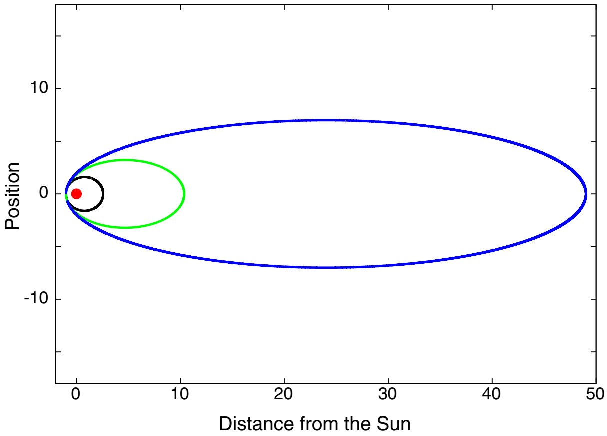

Fig. 2 The orbits of an Earth-like planet. The curves are obtained by Newton's central difference algorithm from the perihelion at $\mathbf{r}(t_0) = (-1, 0)$ with $gM = m = 1$ and with different velocities vy_p. The red filled circle is the position of the Sun, and the three curves are for vy_p= 1.2 (black); 1.3 (green) and 1.4 (blue), respectively

$$\mathbf{r}(t_0 + \delta t) = x(t_0 + \delta t), \, y(t_0 + \delta t) = -r_p + \frac{1}{2}\frac{gM\delta t^2}{r_p^2}, \, \delta t v y_p, \tag{23}$$

and since $x(t_0 + \delta t) = x(t_0 - \delta t)$ due to the time symmetry, the discrete dynamics starts with an energy $E_{\rm disc}(t_0)$ at time t_0=0, which is equal to the constant energy E in the analytic dynamics.

4.1 A shadow Hamiltonian and the functional form of the orbits for the discrete dynamics

The question is: Is there a shadow Hamiltonian for the discrete dynamics of a planet's orbital motion, and if so, what is the functional form of the analytic function for $\tilde{H}$. Since the discrete dynamics for δt going to zero converges to the analytic dynamics with elliptic motion, it is natural to fit an ellipse to the discrete points.

The main investigation is for an Earth-like planet with $vy_p = 1.2$ at $r_p = -1$ and with $gM = m = 1$. The results are given in Table 1 with data for different values of the number n used to integrate one orbit, $n = T(orbit)/\delta t$, where $T(orbit)$ is the orbit time with analytic dynamics (Eq. (13)). The investigation shows several things.

The discrete points are with high precision on an ellipse even for relative few number of integration points n. Figure 3 shows the planet's positions near perihelion and when the position is updated every $\delta t = T(orbit)/365$, or $\approx$ 24 hours for an Earth-like planet. Columns 2 and 3 in the table give the fitted values for the axes and with the rms standard deviations of the fits in column 4, e.g., a deviation of $3. \times 10^{-8}$ corresponds to $\approx$ 3–4 km in the case of planet Earth.

The mean energies, $\langle E_{\rm disc}\rangle$ and $\langle \tilde{E}_{\rm disc}\rangle$, are given in columns 5 and 6. The observed energy fluctuations are decreased by a factor of the order $\approx 10^3$ to 10^5 just by inclusion of the first-order correction (Eq. (22)). Figure 4 shows the energy evolution during tree times in the orbit. The tiny energy variations of the shadow energies are shown in the inset.

The discrete dynamics was obtained for other values of vy_p, $gM.m$ and r_p and confirmed the result, that the discrete dynamics behaves as the analytic. The discrete positions were located on ellipses, and the energies, $\tilde{E}(t_n)$, were almost constant by inclusion of the first-order term (Eq. (22)) in $E_{\rm disc}(t_n)$.

Table 1 Principal axis and discrete energies for $r_p = -1$, $vy_p = 1.2$, $gMm/r_p = -1$ and $\delta t = T(orbit)/n$

n	Major axis	Minor axis	rms	E_{disc}	$\tilde{E}$
365	1.7867062	1.60399	$4. \times 10^{-4}$	-0.27988 ± 3.10^{-5}	-0.2798678 ± 1.10^{-8}
10^3	1.7858364	1.603624	$2. \times 10^{-4}$	-0.279984 ± 3.10^{-6}	$-0.2799823897 \pm 3.^{-10}$
10^4	1.7857156	1.603568016	$2. \times 10^{-7}$	$-0.27999984 \pm 4.10^{-8}$	$-0.2799998239055 \pm 7.10^{-13}$
10^5	1.78571423	1.603567457	$3. \times 10^{-8}$	$-0.2799999985 \pm 4.10^{-10}$	$-0.27999999823913 \pm 1.10^{-14}$
∞	1.78571429	1.603567451	0	-0.28	-0.28

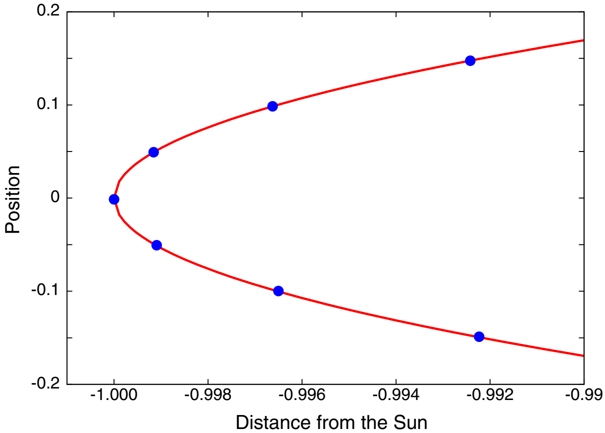

Fig. 3 The discrete positions of an Earth-like planet near perihelion. The discrete positions (blue filled circles) are obtained by Newton's central difference algorithm with $\delta t = T(orbit)/365$, i.e., for an Earth-like planet every 24 h. The full line (red) is an ellipse determined from the 365 discrete points by fitting the axes of an ellipse

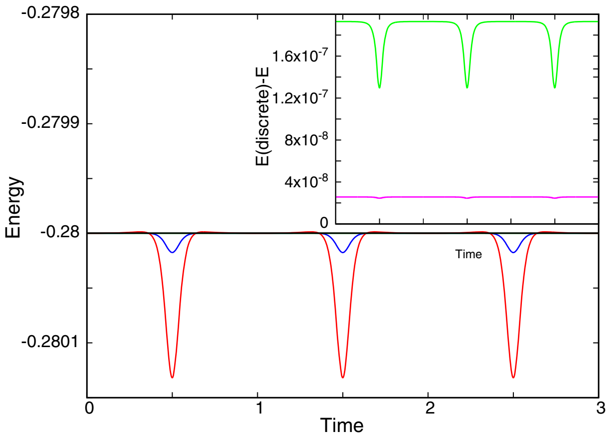

Fig. 4 The energies $E(t_n)$ and $\tilde{E}(t_n)$ for the circulation of a planet tree times in its elliptic orbit. The discrete values are obtained by starting from the aphelion $\mathbf{r}(t_0) = (r_{\max}, 0)$ with $r_{\max}$ and $vy_{\min}$ obtained from $r_p = -1$, $vy_p = 1.2$, $gM = m = 1$ and $\delta t_1 = T(orbit)/365$ and $\delta t_2 = T(orbit)/1000$ ($\approx$ one day and eight hours, respectively). Red: $E(t_n)$ with δt_1; blue: $E(t_n)$ with δt_2; black: $E(analytic) = -0.28$. The inset shows the small energy differences between the corresponding shadow energies, $\Delta E(t_n) = \tilde{E}(t_n) - E(analytic)$. Green: $\Delta E(t_n)$ for δt_1; Magenta: $\Delta E(t_n)$ for δt_2

5 Discussion

The Molecular Dynamics simulations strongly indicate that there exists a shadow Hamiltonian for the discrete Newtonian dynamics of celestial bodies. The existence of a shadow Hamiltonian for the discrete dynamics implies that the positions, obtained by Newton's discrete dynamics, are exact and with the same dynamics invariances as the analytic dynamics: conservation of momenta, angular momenta and total energy. But despite the same dynamic invariances, there is, however, one fundamental difference between the two dynamics. Only the positions and time are variables in the discrete dynamics; the momenta are not.

Newton used the central difference algorithm to derive his second law for classical dynamics, but he never, in *Principia*, calculated a celestial body's positions by using the algorithm. Isaac Newton and Robert Hooke used, however, the geometric implementation (Fig. 1) of the central difference algorithm to construct a celestial body's orbit [22]; but they were of course not aware of, that the discrete dynamics has the same qualities as Newton's analytic dynamics.

The Newtonian analytic dynamics has been questioned. T. D. Lee and coworkers have analysed discrete dynamics in a series of publications. The analysis covers not only classical mechanics [23], but also non-relativistic quantum mechanics and relativistic quantum field theory [24] and Gauge theory and Lattice Gravity [25]. The discrete dynamics is obtained by treating positions and time, but not momenta, as a discrete dynamical variables as in Newton's discrete dynamics. The Newtonian dynamics has also been modified ad hoc by M. Milgrom [26] in order to explain the stability of galaxies.

The indication of the exactness of Newton's discrete dynamics raises the principle question: Which of these two formulations is the correct classical limit law for relativistic quantum dynamics? The momenta and positions in the discrete dynamics are asynchronous as is the case in quantum dynamics, but the difference in the classical limit between the two formulations is, however, immensely small. If the discrete dynamics is the correct formulation, Newton will also be the founder of this dynamics.

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