Short exercises for practicing mathematics used in QM

Some of these were assigned during the year, but are worth doing again.

1. Dirac delta functions
   (a) Problem 2.23
   (b) Problem 2.24
   (c) If a set of N particles in one dimension have positions $x_1, x_2, \ldots, x_N$, write down an expression for the density as a function of position $\rho(x)$ in terms of delta functions.

2. Fourier transforms
   (a) Show that a real function $f(x)$ has a Fourier transform $F(k)$ which satisfies $F(-k) = F(k)^*$
   (b) Show that if $f(x)$ is also even (as well as being real), then $F(k)$ is real and even.
   (c) "Prove" Plancherel's theorem, by working from $\int_{-\infty}^{\infty} e^{ikx} dk = \delta(x)$ (this is the reverse of problem 2.26[2.25])
   (d) Write down the Fourier transform of the density you found in (1c) (call it ρ_k).
   (e) (if you have extra time and want to get even more famliar with Fourier transforms). The convolution of two functions $f(x)$ and $g(x)$, $(f * g)(x)$, is defined by $(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$. Show that the Fourier transform of the convolution is the product of the Fourier transforms of the individual functions (apart from a factor of $\sqrt{2\pi}$).

3. Hermitian operators
   (a) Problem 3.4[3.12, modified]
   (b) Problem 3.5[New]
   (c) Problem 3.7(a)[New]
   (d) Problem 3.13(a)[3.41, modified]
   (e) Prove the Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

4. Harmonic oscillator raising and lowering operators
   (a) Starting from the definitions of a_+ and a_-, and the commutation relation for x and p evaluate the commutator $[a_-, a_+]$.
   (b) Evaulate the expectation values of $a_+^2 a_-^2$ and $a_- a_+^2 a_-$ in the nth harmonic oscillator eigenstate.

5. Operators for three-dimensional quantum mechanics, angular momentum and spin
   (a) Problem 4.1(a)[4.1(a)] (Not much to do really, but make sure you know why the answer is true)
   (b) Derive the angular momentum commutation relations $[L_x, L_y] = i\hbar L_z$ etc using $\vec{L} = \vec{r} \times \vec{p}$
   (c) Derive the commutation relations between L^2 and the components of $\vec{L}$, and between L^2 and $L_\pm$.
   (d) Problem 4.19(a), (b), (c)[4.20 (a),(b),(c)]
   (e) Problem 4.26[4.27]

6. Fermion and boson wavefunctions
   (a) Problem 5.22[5.19 modified] (We didn't do Section 5.4, but this problem does not depend on that material)

7. Matrices as operators
   (a) Problem 3.38(a)[3.39(a)]