Written 4-hour exam in
Thermodynamics and Statistical Mechanics
Thursday 12/01-12, 10.00-14.00

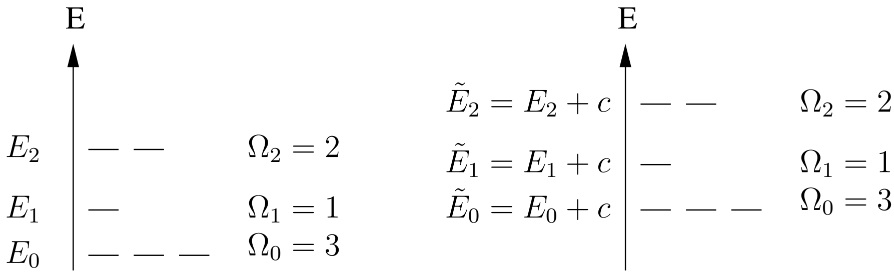
The exam consists of 2 problems on 2 pages altogether. The weights of the problems are given, and within a problem sub-questions have the same weight.

Writing aids and a simple calculator (that is one which can **not** do graphs or symbolic manipulations) are allowed. The “Golden sheet” (3 pages) are included with this exam. No further helping aids are allowed.

Problem 1 (40 %)

Consider a system with $k+1$ discrete energy levels E_n, $n = 0 \dots k$, and let Ω_n be the degeneracy of each energy level (that is the number of micro-states with that energy).
(The left figure gives an example of such a system).

In the following the system is in contact with a heat-bath of a certain temperature and in thermal equilibrium.



Left: Example of system, in this case $k = 2$ as there are 3 energy levels. The number of micro-states for each energy-level is indicated.
Right: System with shifted energy scale, but otherwise same structure of the energy levels.

a) Show that the probability for finding the system at a given energy level is:

$$\mathcal{P}(E_n) = \frac{\Omega_n e^{-E_n\beta}}{\sum_{m=0}^{k} \Omega_m e^{-E_m\beta}} \tag{1}$$

We now additionally consider a system where the energy scale has been changed (shifted) so that $\tilde{E}_n = E_n + c$, where c is some constant (compare left and right figure). In the following variables with a ˜ (a tilde) refer to the system with the shifted energy scale, and variables without the tilde to the original system.

b) Show that the probabilities are unchanged by such a change in energy scale, that is:

$$\tilde{\mathcal{P}}(\tilde{E}_n) = \mathcal{P}(E_n) \tag{2}$$

c) Show that the mean energies of the two systems are related by:

$$\langle \tilde{E} \rangle = \langle E \rangle + c \tag{3}$$

d) Show that the free energies of the two systems are related by:

$$\tilde{F} = F + c \tag{4}$$

e) Show that the entropies of the two systems are related by:

$$\tilde{S} = S \tag{5}$$

and give an expression for the entropy as function of temperature.

f) Calculate the $T \to 0$ and $T \to \infty$ limits of the probability for finding the original system at a given energy (that is find the limits of $\mathcal{P}(E_n)$ for all n's).
Comment on the interpretation of the result.
(*Hint:* Use the results from the previous questions and choose c wisely before trying to evaluate the limits.)

g) Calculate the $T \to 0$ and $T \to \infty$ limits of the entropy for the original system.
Comment on the interpretation of the result.
(*Hint:* Use the results from the previous questions and choose c wisely before trying to evaluate the limits.)

Problem 2 (60 %)

In the following we consider a quasistatic adiabatic volume change, mainly with focus on a mono-atomic ideal-gas.

a) Show that the following relation holds for volume and temperature of a mono-atomic ideal-gas during a quasistatic adiabatic volume change:

$$VT^{3/2} = \text{Constant.} \tag{6}$$

b) Consider a quasistatic adiabatic process in which the volume of an mono-atomic ideal-gas is reduced to 1/3 of the initial volume.
Calculate the final temperature (T_f) and pressure (P_f) in terms of the initial temperature (T_i) and initial pressure (P_i).
Furthermore, calculate the final pressure if the temperature is afterwards reduced to the initial temperature, while holding the volume constant.

c) Show that the following relation between the adiabatic thermal expansion coefficient (α_S), the isochoric specific heat (C_V), the isothermal compressibility (κ_T), and the isobaric thermal expansion coefficient (α_P) holds in *general*:

$$\alpha_S = \frac{C_V \kappa_T}{VT\alpha_P}. \tag{7}$$

d) Show explicitly that equation 7 holds for a mono-atomic ideal-gas by evaluating the left and right hand side for such a gas.

The entropy of a mono-atomic ideal-gas is given by:

$$S = Nk_B \left[\ln\left(\frac{V}{NV_Q}\right) + 5/2 \right], \text{with } V_Q = \left(\frac{h}{\sqrt{2\pi m k_B T}}\right)^3 \tag{8}$$

where m is the mass of the atom.

e) Show *explicitly* that the entropy as given in equation 8 is in fact constant during a quasistatic adiabatic volume change of a mono-atomic ideal-gas (that is a process where equation 6 holds).

End of the exam